MTH 301: Group Theory

Homework IV

(Due 17/10)

1. Show that if G is finite group and p is the smallest prime dividing $|G|$, then any subgroup of index p is normal.

2. Consider the natural action $S_n \curvearrowright \{1, 2, \ldots, n\}$ by permuting its elements.

   (a) Show that the permutation representation of this action is an isomorphism.

   (b) Using this action, show that every permutation $\sigma \in S_n$ has a unique cycle decomposition.

   (c) Show that the order of an element $\sigma \in S_n$ is the least common multiple of the length of the cycles in its unique cycle decomposition.

3. Prove the results from 4.2 (v) of the lesson plan.

4. Consider the self-action $S_n \curvearrowright^c S_n$ by conjugation.

   (a) Show that if $\sigma \in S_n$ is an m-cycle, then each $\tau \in \mathcal{O}_\sigma (= \mathcal{C}_\sigma)$ is also an m-cycle.

   (b) The unique cycle decomposition of a permutation $\sigma \in S_n$ is given by

   $$\sigma = \sigma_1 \sigma_2 \ldots \sigma_{k_\sigma},$$

   where each σ_i is an m_i-cycle, and $\displaystyle\sum_{i=1}^{k_\sigma} m_i = n$. In other words, this decomposition induces a partition of the integer n as follows

   $$n = m_1 + m_2 + \ldots + m_{k_\sigma}.$$

   Show that two permutations of S_n lie in the same conjugacy class if, and only if they induce the same partition of the integer n.

   (c) Using (b), determine the size of each conjugacy class of S_n.

5. Prove the Class Equation stated in 4.2 (vi) of the lesson plan.

6. Let G be a finite group.

   (a) Show that $g \in Z(G)$ if, and only if $|N(g)| = |G|$.

   (b) If $|G| = p^n$, where p is a prime, then show that $Z(G) \neq \{1\}$. [Hint: Use the Class Equation.]

   (c) Show that if $|G| = p^2$, where p is a prime, then G must be abelian.